Hydrogen in Electrodynamics.
V. Dirac-Hydrogen by Means of New Operators

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After a discussion of the one-component Schrödinger (1926) and the four-component Dirac (1928) representation of hydrogen it is shown that the six-component electrodynamic picture is considerably simpler and clearer. The computational effort is reduced to a fraction.

In preparation for the general electrodynamic hydrogen solution we solve the Dirac equation in a simple and clear way, to begin with. We start from the Maxwell-Dirac-isomorphism [1]

$$\left\{\left[\gamma\cdot\nabla + i\frac{\omega}{c}\begin{pmatrix}\varepsilon 1 & 0\\ 0 & \mu 1\end{pmatrix}\right]\Psi = 0\right\} \tag{1}$$

(electronics)

$$\equiv\left\{\left[\gamma\cdot\nabla + i\frac{\omega}{c}\begin{pmatrix}\left(1-\dfrac{\Phi - m_0 c^2}{\hbar\omega}\right)1 & 0\\ 0 & \left(1-\dfrac{\Phi + m_0 c^2}{\hbar\omega}\right)1\end{pmatrix}\right]\Psi = 0\right\}$$

(wave mechanics)

and evaluate the right-hand side of (1) by means of new differential operators derived in [2]. They are:

$$(\partial_1 \pm i\,\partial_2)\, CRP_l^m\, e^{\pm i m\phi} = \frac{C\, e^{\pm i(m+1)\phi}}{2l+1}\left(-R_{,l+1}\, P_{l-1}^{m+1} + R_{,-l}\, P_{l+1}^{m+1}\right), \tag{2}$$

$$\partial_3\, CRP_l^m\, e^{\pm i m\phi} = \frac{C\, e^{\pm i m\phi}}{2l+1}\left[R_{,l+1}(l+m)\, P_{l-1}^m + R_{,-l}(l-m+1)\, P_{l+1}^m\right]. \tag{3}$$

With these operators the solution of the hydrogen problem in Dirac's theory can be obtained much easier and more directly then usually.

After the customary separation of the time dependence by means of

$$\Psi = \psi\, e^{-i\omega t} \tag{4}$$

the Dirac side of (1) reads explicitly

$$\frac{2\pi i}{hc}\left(m_0 c^2 + h\nu + \frac{e_0^2}{r}\right)\psi_1 + (\partial_1 - i\,\partial_2)\psi_4 + \partial_3\psi_3 = 0, \tag{5}$$

$$\frac{2\pi i}{hc}\left(m_0 c^2 + h\nu + \frac{e_0^2}{r}\right)\psi_2 + (\partial_1 + i\,\partial_2)\psi_3 - \partial_3\psi_4 = 0, \tag{6}$$

$$\frac{2\pi i}{hc}\left(-m_0 c^2 + h\nu + \frac{e_0^2}{r}\right)\psi_3 + (\partial_1 - i\,\partial_2)\psi_2 + \partial_3\psi_1 = 0, \tag{7}$$

$$\frac{2\pi i}{hc}\left(-m_0 c^2 + h\nu + \frac{e_0^2}{r}\right)\psi_4 + (\partial_1 + i\,\partial_2)\psi_1 - \partial_3\psi_2 = 0, \tag{8}$$

If, for clarity, we put

$$i\frac{\omega}{c}\,\varepsilon = \frac{2\pi i}{hc}\left(m_0 c^2 + h\nu + \frac{e_0^2}{r}\right) \tag{9}$$

and

$$i\frac{\omega}{c}\,\mu = \frac{2\pi i}{hc}\left(-m_0 c^2 + h\nu + \frac{e_0^2}{r}\right) \tag{10}$$

we now get, instead of (5)–(8), the system

$$i\frac{\omega}{c}\,\varepsilon\psi_1 + (\partial_1 - i\,\partial_2)\,\psi_4 + \partial_3\,\psi_3 = 0, \tag{11}$$

$$i\frac{\omega}{c}\,\varepsilon\psi_2 + (\partial_1 + i\,\partial_2)\,\psi_3 - \partial_3\,\psi_4 = 0, \tag{12}$$

$$i\frac{\omega}{c}\,\mu\psi_3 + (\partial_1 - i\,\partial_2)\,\psi_2 + \partial_3\,\psi_1 = 0, \tag{13}$$

$$i\frac{\omega}{c}\,\mu\psi_4 + (\partial_1 + i\,\partial_2)\,\psi_1 - \partial_3\,\psi_2 = 0. \tag{14}$$

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For the general d'Alembert solution [1] (26) we need the two partial solutions, separated in radial and angular variables,

$$\overset{\pm}{\Psi}_k = \overset{\pm}{\psi}_k\, e^{-i\omega t} = \overset{\pm}{C}_k\, R_k\, P_{l_k}^{\overset{\pm}{m}_k}\, e^{\pm i \overset{\pm}{m}_k \phi}\, e^{-i\omega t} \tag{15}$$

or, respectively, the two amplitudes

$$\overset{\pm}{\psi}_k = \overset{\pm}{C}_k\, R_k\, P_{l_k}^{\overset{\pm}{m}_k}\, e^{\pm i \overset{\pm}{m}_k \phi}\,. \tag{16}$$

We now solve the system (11)–(14) by inserting the ansatz (16) with the lower index, that is

$$\bar{\psi}_k = \bar{C}_k\, R_k\, P_{l_k}^{\bar{m}_k}\, e^{\pm i \bar{m}_k \phi} \tag{17}$$

into (11) and (13). Using (2) and (3) we get

$$\begin{aligned} i\varepsilon\frac{\omega}{c}\,\bar{C}_1\, R_1\, P_{l_1}^{\bar{m}_1}\, e^{-i\bar{m}_1\phi} &+ \bar{C}_4\,\frac{e^{-i(\bar{m}_4+1)\phi}}{2l_4+1}\left(-R_{4,l_4+1}\, P_{l_4-1}^{\bar{m}_4+1} + R_{4,-l_4}\, P_{l_4+1}^{\bar{m}_4+1}\right) \\ &+ \bar{C}_3\,\frac{e^{-i\bar{m}_3\phi}}{2l_3+1}\left[R_{3,l_3+1}\, P_{l_3-1}^{\bar{m}_3}(l_3+\bar{m}_3) + R_{3,-l_3}\, P_{l_3+1}^{\bar{m}_3}(l_3-\bar{m}_3+1)\right] = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} i\mu\frac{\omega}{c}\,\bar{C}_3\, R_3\, P_{l_3}^{\bar{m}_3}\, e^{-i\bar{m}_3\phi} &+ \bar{C}_2\,\frac{e^{-i(\bar{m}_2+1)\phi}}{2l_2+1}\left(-R_{2,l_2+1}\, P_{l_2-1}^{\bar{m}_2+1} + R_{2,-l_2}\, P_{l_2+1}^{\bar{m}_2+1}\right) \\ &+ \bar{C}_1\,\frac{e^{-i\bar{m}_1\phi}}{2l_1+1}\left[R_{1,l_1+1}\, P_{l_1-1}^{\bar{m}_1}(l_1+\bar{m}_1) + R_{1,-l_1}\, P_{l_1+1}^{\bar{m}_1}(l_1-\bar{m}_1+1)\right] = 0, \end{aligned} \tag{19}$$

Here we remember from the derivation [2] of (2) and (3) that the comma-differentiation in the lower index of the radial functions R_k is given by the operator equation

$$R_{k,a} = \left(d_r + \frac{a}{r}\right) R_k \tag{20}$$

according to [2] (17). – From (18)–(19), with

$$\bar{m}_4+1 = \bar{m}_2+1 = \bar{m}_1 = \bar{m}_3 = m+1\,, \tag{21}$$

$$l_3 = l_4 = l_{\mathrm{H}}, \quad R_3 = R_4 = R_{\mathrm{H}}, \tag{22}$$

$$l_1 = l_2 = l_{\mathrm{E}}, \quad R_1 = R_2 = R_{\mathrm{E}} \tag{23}$$

we deduce the system

$$\begin{aligned} (2l_{\mathrm{H}}+1)\, i\varepsilon\frac{\omega}{c}\,\bar{C}_1\, R_{\mathrm{E}}\, P_{l_{\mathrm{E}}}^{m+1} &\\ + \bar{C}_4\left(-R_{\mathrm{H},l_{\mathrm{H}}+1}\, P_{l_{\mathrm{H}}-1}^{m+1} + R_{\mathrm{H},-l_{\mathrm{H}}}\, P_{l_{\mathrm{H}}+1}^{m+1}\right) &\\ + \bar{C}_3\left[R_{\mathrm{H},l_{\mathrm{H}}+1}\, P_{l_{\mathrm{H}}-1}^{m+1}(l_{\mathrm{H}}+m+1)\right. &\\ \left. + R_{\mathrm{H},-l_{\mathrm{H}}}\, P_{l_{\mathrm{H}}+1}^{m+1}(l_{\mathrm{H}}-m)\right] &= 0, \end{aligned} \tag{24}$$

$$\begin{aligned} (2l_{\mathrm{E}}+1)\, i\mu\frac{\omega}{c}\,\bar{C}_3\, R_{\mathrm{H}}\, P_{l_{\mathrm{H}}}^{m+1} &\\ + \bar{C}_2\left(-R_{\mathrm{E},l_{\mathrm{E}}+1}\, P_{l_{\mathrm{E}}-1}^{m+1} + R_{\mathrm{E},-l_{\mathrm{E}}}\, P_{l_{\mathrm{E}}+1}^{m+1}\right) &\\ + \bar{C}_1\left[R_{\mathrm{E},l_{\mathrm{E}}+1}\, P_{l_{\mathrm{E}}-1}^{m+1}(l_{\mathrm{E}}+m+1)\right. &\\ \left. + R_{\mathrm{E},-l_{\mathrm{E}}}\, P_{l_{\mathrm{E}}+1}^{m+1}(l_{\mathrm{E}}-m)\right] &= 0. \end{aligned} \tag{25}$$

The notation l_{E}, l_{H}, R_{E}, R_{H} is arbitrary and can be replaced or interchanged at will; it has been chosen in this way only with regard to a subsequent general electrodynamic hydrogen solution. – In the system (24)–(25) two alternatives are visible: We may either put

$$\bar{C}_4^{\mathrm{I}} = \bar{C}_3^{\mathrm{I}}(l_{\mathrm{H}}^{\mathrm{I}}+m+1), \quad \bar{C}_2^{\mathrm{I}} = -\bar{C}_1^{\mathrm{I}}(l_{\mathrm{E}}^{\mathrm{I}}-m) \tag{26}$$

or

$$\bar{C}_4^{\mathrm{II}} = -\bar{C}_3^{\mathrm{II}}(l_{\mathrm{H}}^{\mathrm{II}}-m), \quad \bar{C}_2^{\mathrm{II}} = \bar{C}_1^{\mathrm{II}}(l_{\mathrm{E}}^{\mathrm{II}}+m+1) \tag{27}$$

For the case (26) we get

$$i\varepsilon\frac{\omega}{c}\,\bar{C}_1^{\mathrm{I}}\, R_{\mathrm{E}}^{\mathrm{I}}\, P_{l_{\mathrm{E}}^{\mathrm{I}}}^{m+1} + \bar{C}_3^{\mathrm{I}}\, R_{\mathrm{H},-l_{\mathrm{H}}^{\mathrm{I}}}^{\mathrm{I}}\, P_{l_{\mathrm{H}}^{\mathrm{I}}+1}^{m+1} = 0, \tag{28}$$

$$-\mu\frac{\omega}{c}\,\bar{C}_3^{\mathrm{I}}\, R_{\mathrm{H}}^{\mathrm{I}}\, P_{l_{\mathrm{H}}^{\mathrm{I}}}^{m+1} + i\,\bar{C}_1^{\mathrm{I}}\, R_{\mathrm{E},l_{\mathrm{E}}^{\mathrm{I}}+1}^{\mathrm{I}}\, P_{l_{\mathrm{E}}^{\mathrm{I}}-1}^{m+1} = 0, \tag{29}$$

and by finally fixing

$$i\bar{C}_1^{\mathrm{I}} = \bar{C}_3^{\mathrm{I}} = \bar{C}^{\mathrm{I}}, \quad l_{\mathrm{H}}^{\mathrm{I}} = l_{\mathrm{E}}^{\mathrm{I}} - 1 = l^{\mathrm{I}} \tag{30}$$

we already have arrived at the first pair of the R-equations of the Dirac theory:

$$\varepsilon\frac{\omega}{c}\, R_{\mathrm{E}}^{\mathrm{I}} + R_{\mathrm{H},-l^{\mathrm{I}}}^{\mathrm{I}} = 0, \tag{31}$$

$$-\mu\frac{\omega}{c}\, R_{\mathrm{H}}^{\mathrm{I}} + R_{\mathrm{E},l^{\mathrm{I}}+2}^{\mathrm{I}} = 0. \tag{32}$$

They possess the hydrogen spectrum.

For the case (27), on the other hand, we get

$$i\varepsilon\frac{\omega}{c}\,\bar{C}_1^{\mathrm{II}}\,R_{\mathrm{E}}^{\mathrm{II}}\,P_{l_{\mathrm{E}}^{\mathrm{II}}}^{m+1}+\bar{C}_3^{\mathrm{II}}\,R^{\mathrm{II}}_{\mathrm{H},\,l_{\mathrm{H}}^{\mathrm{II}}+1}\,P^{m+1}_{l_{\mathrm{H}}^{\mathrm{II}}-1}=0\,,\tag{33}$$

$$-\mu\frac{\omega}{c}\,\bar{C}_3^{\mathrm{II}}\,R_{\mathrm{H}}^{\mathrm{II}}\,P^{m+1}_{l_{\mathrm{H}}^{\mathrm{II}}}+i\,\bar{C}_1^{\mathrm{II}}\,R^{\mathrm{II}}_{\mathrm{E},\,-l_{\mathrm{E}}^{\mathrm{II}}}\,P^{m+1}_{l_{\mathrm{E}}^{\mathrm{II}}+1}=0\,,\tag{34}$$

and by finally fixing

$$i\,\bar{C}_1^{\mathrm{II}}=\bar{C}_3^{\mathrm{II}}=\bar{C}^{\mathrm{II}},\quad l_{\mathrm{H}}^{\mathrm{II}}=l_{\mathrm{E}}^{\mathrm{II}}+1=l^{\mathrm{II}}\tag{35}$$

we have arrived at the second pair of Dirac's R-equations

$$\varepsilon\frac{\omega}{c}\,R_{\mathrm{E}}^{\mathrm{II}}+R^{\mathrm{II}}_{\mathrm{H},\,l^{\mathrm{II}}+1}\;=0\,,\tag{36}$$

$$-\mu\frac{\omega}{c}\,R_{\mathrm{H}}^{\mathrm{II}}+R^{\mathrm{II}}_{\mathrm{E},\,-l^{\mathrm{II}}+1}=0\,,\tag{37}$$

which also possess the hydrogen spectrum.

Turning now to the ansatz (16) with the upper index, that is

$$\overset{+}{\psi}_k=\overset{+}{C}_k\,R_k\,P_{l_k}^{\overset{+}{m}_k}\,e^{\pm i\overset{+}{m}_k\phi}\,,\tag{38}$$

we can proceed analogously. We insert (38) into (12) and (14), again using (2) and (3), and transform the pair of equations obtained in that way by means of

$$\overset{+}{m}_3+1=\overset{+}{m}_1+1=\overset{+}{m}_2=\overset{+}{m}_4=m+1\,,\tag{39}$$

$$l_3=l_4=l_{\mathrm{H}},\quad R_3=R_4=R_{\mathrm{H}},\tag{40}$$

$$l_1=l_2=l_{\mathrm{E}},\quad R_1=R_2=R_{\mathrm{E}}\tag{41}$$

into a pair of equations that doesn't contain the azimuthal angle any more. The, instead of (26) and (27), we have the two alternatives

$$\overset{+}{C}{}_3^{\mathrm{I}}=\overset{+}{C}{}_4^{\mathrm{I}}(l_{\mathrm{H}}^{\mathrm{I}}+m+1),\quad \overset{+}{C}{}_1^{\mathrm{I}}=\overset{+}{C}{}_2^{\mathrm{I}}(l_{\mathrm{E}}^{\mathrm{I}}-m)\tag{42}$$

or

$$\overset{+}{C}{}_3^{\mathrm{II}}=\overset{+}{C}{}_4^{\mathrm{II}}(l_{\mathrm{H}}^{\mathrm{II}}-m),\quad \overset{+}{C}{}_1^{\mathrm{II}}=-\,\overset{+}{C}{}_2^{\mathrm{II}}(l_{\mathrm{E}}^{\mathrm{II}}+m+1)\tag{43}$$

Besides, instead of (30), we now have

$$-i\,\overset{+}{C}{}_2^{\mathrm{I}}=\overset{+}{C}{}_4^{\mathrm{I}}=\overset{+}{C}{}^{\mathrm{I}},\quad l_{\mathrm{H}}^{\mathrm{I}}=l_{\mathrm{E}}^{\mathrm{I}}-1=l^{\mathrm{I}}\tag{44}$$

or

$$-i\,\overset{+}{C}{}_2^{\mathrm{II}}=\overset{+}{C}{}_4^{\mathrm{II}}=\overset{+}{C}{}^{\mathrm{II}},\quad l_{\mathrm{H}}^{\mathrm{II}}=l_{\mathrm{E}}^{\mathrm{II}}+1=l^{\mathrm{II}},\tag{45}$$

after which the two pairs of equations again go over into the two pairs of R-equations (31)–(32) and (36)–(37).

Looking back and summarizing we arrive at the following constants and solutions:

$$\bar{C}_k^{\mathrm{I}}=\bar{C}^{\mathrm{I}}\begin{pmatrix}-i\\ i(l^{\mathrm{I}}-m+1)\\ 1\\ l^{\mathrm{I}}+m+1\end{pmatrix},\quad \bar{C}_k^{\mathrm{II}}=\bar{C}^{\mathrm{II}}\begin{pmatrix}-i\\ -i(l^{\mathrm{II}}+m)\\ 1\\ -(l^{\mathrm{II}}-m)\end{pmatrix};$$

$$\overset{+}{C}{}_k^{\mathrm{I}}=\overset{+}{C}{}^{\mathrm{I}}\begin{pmatrix}i(l^{\mathrm{I}}-m+1)\\ i\\ -(l^{\mathrm{I}}+m+1)\\ 1\end{pmatrix},\quad \overset{+}{C}{}_k^{\mathrm{II}}=\overset{+}{C}{}^{\mathrm{II}}\begin{pmatrix}-i(l^{\mathrm{II}}+m)\\ i\\ l^{\mathrm{II}}-m\\ 1\end{pmatrix};\tag{46}$$

for

$$\bar{\Psi}^{\mathrm{I}}=\bar{C}^{\mathrm{I}}\begin{pmatrix}i\,R_{\mathrm{E}}^{\mathrm{I}}\,P^{m+1}_{l^{\mathrm{I}}+1}\,e^{-i(m+1)\phi}\\ i(l^{\mathrm{I}}-m+1)\,R_{\mathrm{E}}^{\mathrm{I}}\,P^{m}_{l^{\mathrm{I}}+1}\,e^{-im\phi}\\ R_{\mathrm{H}}^{\mathrm{I}}\,P^{m+1}_{l^{\mathrm{I}}}\,e^{-i(m+1)\phi}\\ (l^{\mathrm{I}}+m+1)\,R_{\mathrm{H}}^{\mathrm{I}}\,P^{m}_{l^{\mathrm{I}}}\,e^{-im\phi}\end{pmatrix}e^{-i\omega t},\tag{47}$$

$$\bar{\Psi}^{\mathrm{II}}=\bar{C}^{\mathrm{II}}\begin{pmatrix}-i\,R_{\mathrm{E}}^{\mathrm{II}}\,P^{m+1}_{l^{\mathrm{II}}-1}\,e^{-i(m+1)\phi}\\ -i(l^{\mathrm{II}}+m)\,R_{\mathrm{E}}^{\mathrm{II}}\,P^{m}_{l^{\mathrm{II}}-1}\,e^{-im\phi}\\ R_{\mathrm{E}}^{\mathrm{II}}\,P^{m+1}_{l^{\mathrm{II}}}\,e^{-i(m+1)\phi}\\ -(l^{\mathrm{II}}-m)\,R_{\mathrm{H}}^{\mathrm{II}}\,P^{m}_{l^{\mathrm{II}}}\,e^{-im\phi}\end{pmatrix}e^{-i\omega t},\tag{48}$$

$$\overset{+}{\Psi}{}^{\mathrm{I}}=\overset{+}{C}{}^{\mathrm{I}}\begin{pmatrix}i(l^{\mathrm{I}}-m+1)\,R_{\mathrm{E}}^{\mathrm{I}}\,P^{m}_{l^{\mathrm{I}}+1}\,e^{im\phi}\\ i\,R_{\mathrm{E}}^{\mathrm{I}}\,P^{m+1}_{l^{\mathrm{I}}+1}\,e^{i(m+1)\phi}\\ -(l^{\mathrm{I}}+m+1)\,R_{\mathrm{H}}^{\mathrm{I}}\,P^{m}_{l^{\mathrm{I}}}\,e^{im\phi}\\ R_{\mathrm{H}}^{\mathrm{I}}\,P^{m+1}_{l^{\mathrm{I}}}\,e^{i(m+1)\phi}\end{pmatrix}e^{-i\omega t},\tag{49}$$

$$\overset{+}{\Psi}{}^{\mathrm{II}}=\overset{+}{C}{}^{\mathrm{II}}\begin{pmatrix}-i(l^{\mathrm{II}}+m)\,R_{\mathrm{E}}^{\mathrm{II}}\,P^{m}_{l^{\mathrm{II}}-1}\,e^{im\phi}\\ i\,R_{\mathrm{E}}^{\mathrm{II}}\,P^{m+1}_{l^{\mathrm{II}}-1}\,e^{i(m+1)\phi}\\ (l^{\mathrm{II}}-m)\,R_{\mathrm{H}}^{\mathrm{II}}\,P^{m}_{l^{\mathrm{II}}}\,e^{im\phi}\\ R_{\mathrm{H}}^{\mathrm{II}}\,P^{m+1}_{l^{\mathrm{II}}}\,e^{i(m+1)\phi}\end{pmatrix}e^{-i\omega t}.\tag{50}$$

[1] H. Sallhofer, Z. Naturforsch. **43a,** 1039 (1988).

[2] H. Sallhofer, Z. Naturforsch. **44a,** 167 (1989).